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Linear Operators Which Preserve Combinatorial Orthogonality

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Dedicated to Marvin Marcus

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ABSTRACT

A $(1, -1, 0)$ matrix is said to be a potentially sign-orthogonal sign pattern if the inner product of any pair of distinct rows or columns equals zero or contains terms that are positive and terms that are negative. A potentially orthogonal matrix is a matrix such that some matrix with the same $(+, -, 0)$ sign pattern is an orthogonal matrix. Evidently all potentially orthogonal $(1, -1, 0)$ matrices are potentially sign-orthogonal sign patterns, but not conversely. In this article we characterize the linear operators on the $n \times n$ real matrices that map the set $\mathscr{X}$ of matrices with potentially sign-orthogonal sign patterns to itself and the complement of $\mathscr{X}$ to itself. We also show that the same operators are the only ones that map the set $\mathscr{Y}$ of potentially orthogonal matrices to itself and the complement of $\mathscr{Y}$ to itself.

1. INTRODUCTION

Recently there has been considerable interest in what is called *structured* matrix theory, *qualitative* matrix theory, or *combinatorial* matrix theory by

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various researchers. These terms are used to indicate that area of matrix theory where only the sign of an entry or whether it is zero or nonzero is considered. The magnitude of any entry is not of concern in structured matrix theory. One way to determine properties of a set of matrices is to investigate the structure of linear operators which preserve the set. This approach was begun by Frobenius in 1897 [4] when he investigated linear operators that preserve the determinant. In 1959, Marcus and Moyls rekindled interest in this approach by characterizing linear preservers of the rank function [5] and the preservers of the set of matrices of rank 1 [6]. In this paper we investigate the linear operators preserving sets of matrices defined by the possible orthogonal properties of their sign patterns. This complements the study of preservers of the orthogonal group by Botta and Pierce [3].

Throughout this article, all matrices are square real matrices. We denote the set of real $n \times n$ matrices by $\mathscr{M}_n$ and suppress the subscript unless the order of the matrices is needed to avoid confusion.

It is possible to glean some information about the possibility of a matrix being orthogonal by studying the sign pattern $(+, -, 0)$ formed by the respective entries of the matrix. For example, no 4×4 real matrix with sign pattern

$$\begin{bmatrix} 0 & + & + & + \\ + & 0 & + & + \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \tag{1}$$

can be orthogonal, because the inner product of the first and second rows cannot be zero.

This notion is made rigorous by defining the *sign pattern* of a matrix $A = (a_{ij})$ as the $(1, -1, 0)$ matrix

$$s(A) = (s_{ij}), \qquad \text{where} \quad s_{ij} = \begin{cases} 1 & \text{if} \quad a_{ij} > 0, \\ 0 & \text{if} \quad a_{ij} = 0, \\ -1 & \text{if} \quad a_{ij} < 0. \end{cases}$$

When specifying a sign pattern of a matrix we will usually write "+" for "+1", "−" for "−1", and "0" for "0".

If P is a property of some matrices, for example, that of being orthogonal, nonsingular, stable, or Hermitian, we say that a sign pattern *requires* property P if every matrix with that sign pattern must have property P. A sign pattern *allows* property P if some matrix with that sign pattern has property P. For example, a sign pattern of +'s on the main diagonal and 0's off requires nonsingularity and allows orthogonality. Evidently, no sign pattern can require orthogonality, since the norm of any nonzero row cannot be required to be 1 by its sign pattern alone. A sign pattern matrix is said to be *potentially orthogonal* (PO) if it allows orthogonality. For more discussion about potentially orthogonal matrices, see [1].

We say the matrix A *dominates* the matrix B if A and B have the same orders and $a_{ij} = 0$ implies $b_{ij} = 0$ for all i and j.

Two sign pattern vectors, $\mathbf{s}$ and $\mathbf{t}$, are *potentially orthogonal* if there exist vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^r$ with sign patterns $\mathbf{s}$ and $\mathbf{t}$ respectively such that $\mathbf{u} \cdot \mathbf{v} = 0$. Equivalently, $\mathbf{s}$ and $\mathbf{t}$ are potentially orthogonal if and only if either

$$\{s_i t_i : i = 1, \dots, n\} = \{0\} \quad \text{or} \quad \{s_i t_i : i = 1, \dots, n\} \supseteq \{-1, 1\}.$$

A matrix is *potentially sign-row-orthogonal* (PSRO) if it dominates a permutation matrix and its sign pattern matrix has pairwise potentially orthogonal rows. *Potentially sign-column-orthogonal* (PSCO) is defined in a parallel fashion. A matrix is *potentially sign-orthogonal* (PSO) if it is both PSRO and PSCO. The sign pattern (1) is PSCO but not PSRO, and therefore neither PSO nor PO.

It is clear that any potentially orthogonal matrix is PSO, but the converse is not true. Consider the sign pattern matrix

$$\begin{bmatrix} - & + & + & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ + & + & - & 0 & 0 & 0 \\ + & + & + & - & + & + \\ + & + & + & + & - & + \\ + & + & + & + & + & - \end{bmatrix}.$$

It is easily seen to be PSRO and PSCO, and hence PSO. However, no orthogonal matrix can have this sign pattern. A current area of interest is to determine whether every $(1, -1)$ matrix which is PSO allows orthogonality. Neither proof nor counterexample is known. Also, there is no known example of an irreducible PSO sign pattern matrix which does not allow orthogonality.

We say a linear operator $\mathbf{T}$ on $\mathcal{M}$ *preserves* PO (respectively, PSO, PSRO, PSCO) if $\mathbf{T}(A)$ is PO (respectively, PSO, PSRO, PSCO) whenever A is. Also $\mathbf{T}$ *strongly preserves* PO (respectively, PSO, PSRO, PSCO) if $\mathbf{T}(A)$ is PO (respectively, PSO, PSRO, PSCO) if and only if A is PO (respectively, PSO, PSRO, PSCO). In this article we investigate the set of linear operators on $\mathcal{M}$ which strongly preserve PO, PSO, PSRO and PSCO.

Let $\circ$ denote the *Hadamard* product of two matrices. That is, $A \circ B = C$ if $c_{ij} = a_{ij} b_{ij}$. Some authors call this the *Schur* product.

We show that $\mathbf{T}: \mathscr{M} \to \mathscr{M}$ strongly preserves PO if and only if $\mathbf{T}$ strongly preserves PSO if and only if there exist permutation matrices P and Q in $\mathscr{M}$, diagonal $(1, -1)$ matrices C and D in $\mathscr{M}$, and M in $\mathscr{M}$, all of whose entries are strictly positive, such that either

$$\mathbf{T}(X) = PC(X \circ M)DQ \qquad \text{for all } X \text{ in } \mathscr{M} \tag{2a}$$

or

$$\mathbf{T}(X) = [PC(X \circ M)DQ]^t \qquad \text{for all } X \text{ in } \mathscr{M}. \tag{2b}$$

We also show that for $n \geqslant 4$, $\mathbf{T}$ strongly preserves PSRO or PSCO if and only if (2a) holds.

One would hope that all operators that preserve PO, PSO, PSRO, or PSCO also preserve the set strongly; however, this is not the case. The operators in (2) are nonsingular, but as the following example shows, there are some operators that preserve PO, PSO, PSRO, and PSCO that are singular and do not strongly preserve PO, PSO, PSRO, or PSCO.

EXAMPLE 1.1. Let $\mathbf{L}_2$ denote the operator on $\mathscr{M}_2$ defined by

$$\mathbf{L}_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

and let $\mathbf{L}_4$ denote the operator on $\mathscr{M}_4$ defined by

$$\mathbf{L}_4\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ p & q & r & s \end{bmatrix}\right) = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}.$$

Both $\mathbf{L}_2$ and $\mathbf{L}_4$ preserve PO, PSO, PSRO, and PSCO. In fact, they map any matrix with a nonzero first row to a PO matrix, and clearly these are singular operators. If in addition, $U \in \mathscr{M}_2$ and $V \in \mathscr{M}_4$ are orthogonal matrices then $\mathbf{L}_2(U)$ and $\mathbf{L}_4(V)$ are orthogonal. Therefore, they are also examples of singular operators which preserve the group of orthogonal matrices as well. A similar example exists for 8×8 matrices. However, no other examples are known.

The above example illustrates the need for a hypothesis like "strong" as in our results. Except for $n = 2$, 4, and 8, this restriction may not be necessary.

Botta and Pierce [3] have investigated the preservers of an orthogonal group. Their results require the hypothesis that $\mathbf{T}$ be nonsingular. We use the hypothesis that $\mathbf{T}$ is a strong preserver and show in Lemma 2.1 that $\mathbf{T}$ is then also nonsingular. Our proof of Lemma 2.2 requires that $\mathbf{T}$ be both nonsingular and a strong preserver. We conjecture that $\mathbf{T}$ strongly preserves PO (respectively, PSO, PSRO, PSCO) if and only if $\mathbf{T}$ is nonsingular and preserves PO (respectively, PSO, PSRO, PSCO).

2. PRELIMINARY RESULTS

It is easily checked that if a linear operator satisfies (2a) then $\mathbf{T}$ strongly preserves PO, PSO, PSRO, and PSCO. If $\mathbf{T}$ satisfies (2b) then $\mathbf{T}$ strongly preserves PO and PSO.

In what follows, we will use the following sign pattern matrix which is PO and, hence, PSO, PSRO, and PSCO:

$$\begin{bmatrix} + & + & + & \cdots & + & + \\ + & - & + & \cdots & + & + \\ + & + & - & \cdots & + & + \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ + & + & + & \cdots & - & + \\ + & + & + & \cdots & + & - \end{bmatrix} = (J_n - I_n) + [1 \oplus (-I_{n-1})], \quad (3)$$

where J_n is the matrix of all 1's and I_n is the identity matrix. The $n \times n$ orthogonal matrix $U_n = (u_{ij})$ defined by $u_{11} = 2/(n+1)$, $u_{i1} = u_{1i} = \sqrt{n+3}/(n+1)$ for $2 \leqslant i \leqslant n$, $u_{ii} = -n/(n+1)$ for $2 \leqslant i \leqslant n$, and $u_{ij} = u_{ji} = 1/(n+1)$ for $2 \leqslant i < j \leqslant n$ proves that the sign pattern (3) is PO. A *cell* is a matrix with exactly one nonzero entry and it equals 1. If the nonzero entry of a cell is in the (i, j) location, we denote the cell E_{ij}. The matrix tE_{ij} for some nonzero t in $\mathbb{R}$ is called a *weighted cell.*

LEMMA 2.1. *If* $\mathbf{T}$ *is a linear operator on* $\mathcal{M}$ *which strongly preserves PO, PSO, PSRO, or PSCO, then* $\mathbf{T}$ *is nonsingular.*

Proof. Suppose $\mathbf{T}$ strongly preserves PO, and that $\mathbf{T}(X) = O$ for a nonzero matrix X. Since operators of the type (2a) strongly preserve PO, we can assume that $x_{nn} < 0$. Let α be a real number strictly larger than the absolute value of any entry in X, and let B be the matrix with sign pattern (3) and the absolute value of each entry equal to α. Let $A = B + \alpha E_{nn}$. That is, A is the same matrix as B except that the (n, n) entry of A is zero.

Further, by the choice of α, $A + X$ has sign pattern (3), so that $A + X$ is PO. However, the inner product of the first and last rows or columns of A is always positive; hence A is not PO. Since $\mathbf{T}(X) = O$, $\mathbf{T}(A + X) = \mathbf{T}(A)$, contradicting that $\mathbf{T}$ strongly preserves PO. Thus $\mathbf{T}$ is nonsingular.

By replacing PO with PSO, PSRO, and PSCO in turn in the above argument, the lemma is established, since in each case the matrix $A + X$ is both PSRO and PSCO while A is neither. ∎

LEMMA 2.2. *If* $\mathbf{T}$ *strongly preserves PO, PSO, PSRO, or PSCO, then* $\mathbf{T}$ *is a bijection on the set of weighted cells.*

Proof. Suppose $\mathbf{T}$ strongly preserves PO, and let k, l be fixed integers, $1 \leqslant k, l \leqslant n$. We must show that $\mathbf{T}(E_{kl}) = tE_{rs}$ for some $1 \leqslant r, s \leqslant n$ and nonzero real number t. By virtue of the fact that every operator of the type (2a) strongly preserves PO, we may assume that $(k, l) = (n, n)$. Let $x_{11} = 1$, and choose $x_{12} > 0$ such that for every pair (p, q) such that the (p, q) entry of $\mathbf{T}(E_{11})$ or $\mathbf{T}(E_{12})$ is nonzero, we have that the (p, q) entry of $\mathbf{T}(E_{11} + x_{12}E_{12})$ is nonzero. Next, choose $x_{13} > 0$ such that for every pair of (p, q) such that the (p, q) entry of $\mathbf{T}(E_{11})$ or $\mathbf{T}(E_{12})$ or $\mathbf{T}(E_{13})$ is nonzero, we have that the (p, q) entry of $T(E_{11} + x_{12}E_{12} + x_{13}E_{13})$ is nonzero. Continue in this way choosing $x_{ij} > 0$ where $i \neq j$, $1 \leqslant i, j \leqslant n$, and $x_{ii} < 0$, $1 < i < n$. Let $x_{nn} = 0$. We have thus chosen X with sign pattern (3), except that $x_{nn} = 0$. Further, we have chosen S such that if the (p, q) entry of $\mathbf{T}(E_{ij})$, for any pair $(i, j) \neq (n, n)$, is nonzero, then the (p, q) entry of $\mathbf{T}(X)$ is nonzero.

Choose t_{nn} so that $0 < t_{nn} < |u/v|$, where u is the smallest in absolute value of all nonzero entries of $\mathbf{T}(X)$ and v is the largest in absolute value of all entries in $\mathbf{T}(E_{nn})$. By Lemma 2.1, u exists and $v \neq 0$. Now, by the choice of t_{nn}, every nonzero entry in $\mathbf{T}(X)$ has the same sign as that entry of $\mathbf{T}(X) \pm t_{nn}\mathbf{T}(E_{nn})$. But $X + t_{nn}E_{nn}$ is not PO while $X - t_{nn}E_{nn}$ is PO. Thus, there are some entries which are zero in $\mathbf{T}(X)$, and thus zero in $\mathbf{T}(E_{rs})$ for all $(r, s) \neq (n, n)$, but nonzero in $\mathbf{T}(E_{nn})$. Since k, l were chosen arbitrarily, the above process gives for each pair, (k, l), a nonempty subset, I_{kl}, of ordered pairs (p, q) such that the (p, q) entry of $\mathbf{T}(E_{kl})$ is nonzero but the (p, q) entry of $\mathbf{T}(E_{rs})$ is zero for all $(r, s) \neq (k, l)$. That is, $I_{kl} \cap I_{rs} = \varnothing$ and $\bigcup_{k,l=1}^{n} I_{kl} \subseteq \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$. It follows that I_{kl} is a singleton for each pair (k, l). Define Σ, mapping the set of ordered pairs of integers between 1 and n to itself, $\Sigma(k, l) = (p, q)$ if $I_{kl} = \{(p, q)\}$. Then Σ is a bijection, and the lemma is established for $\mathbf{T}$ strongly preserving PO.

By replacing PO with PSO, PSRO, and PSCO in turn in the above argument, the lemma follows. ∎

The *term rank* of a matrix A is the smallest number of lines (a *line* is either a row or a column) which contain all the nonzero entries of A. In the sequel we shall use the following lemma due to Beasley and Pullman.

LEMMA 2.3 [2, Corollary 3.1.2]. *Suppose that* $\mathbf{T}$ *is a nonsingular operator on* $\mathcal{M}$. *Then* $\mathbf{T}$ *preserves the set of matrices of term rank* 1 *if and only if* $\mathbf{T}$ *is one of or a composition of some of the following operators*:

(*a*) $X \to X^t$.

(*b*) $X \to PXQ$ *for some fixed but arbitrary permutation matrices* P *and* Q *in* $\mathcal{M}$.

(c) $X \to X \circ A$ *for some fixed but arbitrary matrix* A *in* $\mathcal{M}$ *with all nonzero entries.*

LEMMA 2.4. *If* $\mathbf{T}$ *strongly preserves PO, PSO, PSRO, or PSCO, then* $\mathbf{T}$ *preserves the set of matrices of term rank* 1.

Proof. Suppose that $\mathbf{T}$ strongly preserves PO (respectively, PSO, PSRO, PSCO), and let $1 \leqslant i \leqslant n$. Let $R_i = E_{i1} + E_{i2} + \cdots + E_{in}$ and let $C_i = E_{1i} + E_{2i} + \cdots + E_{ni}$. (Note that every term rank 1 matrix is dominated by some R_i or C_i.) Since $\mathbf{T}$ is bijective on the set of weighted cells by Lemma 2.2, $\mathbf{T}(R_i)$ and $\mathbf{T}(C_i)$ have exactly n nonzero entries. If $\mathbf{T}(R_i)$ or $\mathbf{T}(C_i)$ is not a line then two collinear cells, say E_{ij} and E_{ik} have noncolinear images, say $\mathbf{T}(E_{ij}) = t_j E_{pq}$ and $\mathbf{T}(_{ik}) = t_k E_{rs}$ with $p \neq r$ and $q \neq s$.

Let X be a matrix with exactly $n - 2$ nonzero entries such that $\mathbf{T}(E_{ij} + E_{ik} + X)$ is a weighted permutation matrix. This is possible because $\mathbf{T}$ is bijective on the set of cells. Then $\mathbf{T}(E_{ij} + E_{ik} + X)$ is PO and hence PSO, PSCO, and PSRO, but since $E_{ij} + E_{ik} + X$ contains a zero row, it is neither PO, PSO, PSRO, nor PSCO, since it does not dominate a permutation matrix. This contradicts the fact that $\mathbf{T}$ strongly preserves PO (respectively, PSO, PSRO, PSCO). Thus $\mathbf{T}$ preserves the set of term rank 1 matrices and the lemma is proved. ∎

Any matrix with sign pattern (1) gives an example of a 4×4 matrix which can be extended to an $n \times n$ matrix for each $n \geqslant 4$, which is PSCO but not PSRO and hence neither PSO nor PO. Our next lemma shows that no such examples exist for $n \leqslant 3$.

LEMMA 2.5. *Suppose* A *is a sign pattern in* $\mathcal{M}_n$ *with* $n \leqslant 3$. *The following are equivalent*:

(*a*) A *is PO*;

(b) A *is PSO*;

(c) A *is PSRO*; *and*

(d) A *is PSCO*.

Proof. By considering a few cases, it can be shown that each PSRO sign pattern in $\mathcal{M}_n$, for $n \leqslant 3$, can be obtained from one of the following eight sign patterns by permuting rows and/or columns and/or multiplying rows and/or columns by -1:

$$[+],$$

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & - \end{bmatrix},$$

$$\begin{bmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & - & 0 \\ 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & - \\ + & - & 0 \end{bmatrix},$$

$$\begin{bmatrix} + & + & + \\ + & + & - \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & - \\ + & - & - \end{bmatrix}.$$

And clearly, every sign pattern obtained from these eight in the above manner is PSRO. Each of these sign patterns is symmetric; thus a sign pattern is PSCO if and only if it can be obtained from one of these eight in the same manner. Since A is PSO if and only if A is PSRO and PSCO, a sign pattern is PSO if and only if it can be obtained from one of these eight patterns as above. Thus (b), (c), and (d) are equivalent.

Finally, since every PO sign pattern is PSO and each of the eight sign patterns above is PO, (a) is equivalent to the others. ■

3. THE MAIN RESULT

In the following theorem, we shall refer to the following four types of linear operators:

I. X → PXQ *for some fixed but arbitrary permutation matrices P and Q;*

II. $X \to RXS$ for some fixed but arbitrary signature matrices R and S (diagonal matrices for whose diagonal entries are $+1$ or -1);

III. $X \to X \circ M$ for some fixed but arbitrary matrix M with all positive entries;

IV. $X \to X^t$.

THEOREM 3.1. *Let $\mathbb{T}$ be a linear operator on $\mathcal{M}_n(\mathbb{R})$.*

A. If $n > 3$, then $\mathbb{T}$ strongly preserves PSRO iff $\mathbb{T}$ strongly preserves PSCO iff $\mathbb{T}$ is one of or a composition of some of the operators of type I, II, or III.

B. *If* $n \leqslant 3$, *then* $\mathbf{T}$ *strongly preserves PSRO iff* $\mathbf{T}$ *strongly preserves PSCO iff* $\mathbf{T}$ *strongly preserves PSO iff* $\mathbf{T}$ *strongly preserves PO iff* $\mathbf{T}$ *is one of or a composition of some of the operators of type I, II, III, or IV.*

C. $\mathbf{T}$ *strongly preserves PSO iff* $\mathbf{T}$ *strongly preserves PO iff* $\mathbf{T}$ *is one of or a composition of some of the operators of type I, II, III, or IV.*

Proof. An easy check shows that operators of types I, II, and III strongly preserve PSRO, PSCO, PSO, and PO. In addition, it is easily seen that the operators of type IV strongly preserve PO and PSO. In case $n \leqslant 3$, Lemma 2.5 implies that operators of type I, II, III, and IV strongly preserve PSRO and PSCO. This proves necessity in parts A, B, and C of the theorem.

Suppose $n > 3$ and $\mathbf{T}$ strongly preserves PSRO or PSCO. Let A be any 4×4 matrix with sign pattern (1). The matrix $A \oplus I_{n-4}$ is PSCO but not PSRO, so the transpose operator (type IV) does not strongly preserve PSRO. Similarly, the transpose operator does not strongly preserve PSCO. Applying this and Lemma 2.4 to Lemma 2.3, we have that $\mathbf{T}$ is in the semigroup of operators generated by operators of type I and operators of the form $X \to X \circ W$ for some W all of whose entries are nonzero.

Suppose that $\mathbf{T}$ strongly preserves PO or PSO. Applying Lemma 2.4 to Lemma 2.3, we have that $\mathbf{T}$ is in the group of operators generated by operators of type I and IV and operators of the form $X \to X \circ W$ as above.

Suppose that $\mathbf{T}$ is an operator of the form $X \to X \circ W$ where W has all nonzero entries, and suppose that $\mathbf{T}$ strongly preserves PO, PSO, PSCO, or PSRO. To finish sufficiency in parts A, B, and C of the theorem we need only show that $\mathbf{T}$ is a composition of operators of types II and III.

Let R be the diagonal matrix whose diagonal entries are $r_{ii} = w_{i1}/|w_{i1}|$, and let S be the diagonal matrix whose diagonal entries are $s_{ii} = r_{11}w_{1i}/|w_{1i}|$, for all i. Note that each diagonal entry of both R and S is either $+1$ or -1. Let $\mathbf{T}_1$ be the operator $X \to RXS$, and let $\mathbf{T}_2$ be the operator $X \to X \circ M$, where $M = RWS$. Then $\mathbf{T}(X) = \mathbf{T}_1(\mathbf{T}_2(X))$. We only need show that M has all positive entries to show that $\mathbf{T}$ is a composition of operators of type II and III.

By the definition of R and S we have that m_{1j} and m_{i1} are positive for all i, j. Suppose that m_{kl} is negative for some k and l. By permuting rows and columns, we may assume that $k = l = 2$ without losing $m_{1j}, m_{i1} > 0$. Let A be the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \oplus I_{n-2}.$$

Then $\mathbf{T}_2(A)$ is neither PSRO nor PSCO, and hence neither PSO nor PO, while A is PO, and hence PSO, PSRO, and PSCO. Thus, $\mathbf{T}(A)$ is neither PO, PSO, PSRO, nor PSCO, while A is all of these. This contradiction implies that M has only positive entries, and the theorem follows. ∎

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